Unions of Perfect Matchings in r-graphs

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Abstract

An r-regular graph is said to be an r-graph if $|\partial(X)| \geq r$ for each odd set $X \subseteq V(G)$, where $|\partial(X)|$ denotes the set of edges with precisely one end in X. Note that every connected bridgeless cubic graph is a 3-graph. The Berge Conjecture states that every 3-graph G has five perfect matchings such that each edge of G is contained in at least one of them. Likewise, generalization of the Berge Conjecture asserts that every r-graph G has $2r-1$ perfect matchings that covers each $e \in E(G)$ at least once. A natural question to ask in the light of the Generalized Berge Conjecture is that what can we say about the proportion of edges of an r-graph that can be covered by union of t perfect matchings? In this paper we provide a lower bound to this question. We will also present a new conjecture that might help towards the proof of the Generalized Berge Conjecture.

1 Introduction

Graphs in this paper are simple unless otherwise specified. Let G be a graph, $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. A perfect matching of G is a set of edges, $M \subseteq E(G)$, such that each vertex in G is incident with exactly one edge in M.

A cubic graph is one in which every vertex is incident with exactly three edges. An edge is called a bridge if its deletion increases the graph's number of components and a graph is bridgeless if it contains no bridge. One of the earliest results in graph theory due to Petersen ([9]) states that every bridgeless cubic graph has a perfect matching. Applying the Tutte's theorem ([11]) which states that a graph G has a perfect matching if and only if for every $X \subseteq V(G)$, $G - X$ has at most $|X|$ components with odd number of vertices, we have that every edge of a bridgeless cubic graph G is contained in a perfect matching of G. So the question is: what is the minimum number of perfect matchings so that every edge of a bridgeless cubic graph is covered by the union of them? In the early seventies Berge conjectured that this number is 5:

Conjecture 1.1 (Berge Conjecture). *Every bridgeless cubic graph has five perfect matchings such that each edge of G is contained in at least one them.*

Another well-known conjecture, which attributed to Berge in [10] but published first by Fulkerson in [2] states that every bridgeless cubic graph contains a family of six perfect matchings covering each edge of the graph exactly twice.

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Conjecture 1.2 (Berge-Fulkerson Conjecture)**.** *Every bridgeless cubic graph G has a family of six perfect matchings such that each edge of G contained in precisely two of them.*

A set of six perfect matchings that satisfies Conjecture 1.2 is called a Fulkerson cover of G. Any five perfect matchings of a Fulkerson cover of a graph G covers $E(G)$. Hence Conjecture 1.2 implies Conjecture 1.1. It was proved by Mazzuoccolo in [5] that these two conjectures are actually equivalent.

An r-regular graph G is said to be an r-graph if $|\partial(X)| \geq r$ for each odd set $X \subseteq V(G)$, where $\partial(X)$ denotes the set of edges with precisely one end in X. Notice that every bridgeless cubic graph is a 3-graph.

For a fixed positive integer r, let $m_t^r(G)$ be the maximum fraction of the edges in an r-graph G that can be covered by t perfect matchings, and m_t^r be the infimum of all $m_t^r(G)$ over all r-graphs, that is

$$m_t^r(G) = \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}, \text{ and } \quad m_t^r = \inf_G m_t^r(G).$$

The above notation was introduced by Mazzuoccolo in [?]. (The referee asked us to give credit to Mazzuoccolo).

Let P denote the Petersen Graph. Then one can easily observe that $m_1^3(P) = \frac{1}{3}, m_2^3(P) = \frac{3}{5}$, $m_3^3(P) = \frac{4}{5}$, $m_4^3(P) = \frac{14}{15}$, $m_5^3(P) = 1$. The following conjecture given by Patel in [8] is the refinement of Conjecture 1.1.

Conjecture 1.3 ([8], Patel)**.** $m_t^3 = m_t^3(P)$ *for* $1 \leq t \leq 5$, *where* P *is the Petersen Graph.*

Conjecture 1.3 was proved by Kaiser, Král, and Norine in [4] for the case $t = 2$ and they also showed that $\frac{27}{35} \leq m_3^3 \leq \frac{4}{5}$. The exact values for m_3^3 and m_4^3 are still unknown. Moreover the bounds for m_4^3 was predicted in [4] as $\frac{55}{63} \leq m_4^3 \leq \frac{14}{15} = m_4^3(P)$ and the lower bound explicitly proved in [7].

They showed that $\frac{27}{35} \leq m_3^3 \leq \frac{4}{5}$ and predicted that $\frac{55}{63} \leq m_4^3 \leq \frac{14}{15} = m_4^3(P)$? Mazzuoccolo in [7] confirmed the lower bound. [Please see whether this is better]

A natural generalizations of the Conjectures 1.1 and 1.2 to r-graphs given by Seymour in [10] are as follows:

Conjecture 1.4 (Generalized Berge Conjecture)**.** *Every r-graph has $2r-1$ perfect matchings such that each edge is contained at least one of them. That is* $m_{2r-1}^r = 1$.

Conjecture 1.5 (Generalized Berge-Fulkerson Conjecture)**.** *Every r-graph has $2r$ perfect matchings such that each edge is contained in exactly two of them.*

In [6] Mazzuoccolo showed that Conjecture 1.4 and Conjecture 1.5 are equivalent and the value $2r-1$ in Conjecture 1.4 is best possible.

In this paper we are concerning about the values m_t^r for $r \geq 3$ and we prove a recursive lower bound shown in the following theorem.

Theorem 1. *For any fixed integer $r \geq 3$, let $a_0^r, a_1^r, \ldots$ be a sequence of rational numbers satisfying $a_0^r = 0$ and $a_t^r = a_{t-1}^r + (1 - a_{t-1}^r)\frac{2+(t-1)(r^2-r-4)}{2r+(t-1)(r^3-r^2-6r+4)}$. Then for any positive integer t, we have $m_t^r \geq a_t^r$.*

Remark 1. *Jin and Steffen in [3] claimed a better lower bound for m_t^r than in Theorem 1. In a private communication, Jin confirmed that their bound can not be verified due to some gaps in their proof.* The referee(s) asked us to provide more details, something as the below may help. In their proof (page _, line(s) _), they forgot the condition that $|X|$ is odd...

Note that the special case when $r = 3$ of Theorem 1,

$$m_t^3 \geq a_t^3 = a_{t-1}^3 + (1 - a_{t-1}^3)\frac{2 + 2(t-1)}{6 + 4(t-1)} = a_{t-1}^3 + (1 - a_{t-1}^3)\frac{t}{2t+1},$$

agrees with the bound given by Mazzuoccolo in [7].

Recall that Seymour conjectured that $m_{2r-1}^r = 1$ for any $r \geq 3$. By direct calculation using Theorem 1, we have the following lower bounds for m_{2r-1}^r for some values of r;

r	a_{2r-1}^r
3	0.930736
4	0.897367
5	0.885256
6	0.878973
7	0.875227
100	0.864721
1000	0.864665

In Section 2, we will introduce the Edmonds' Perfect Matching Polytope Theorem which is the main tool we use in the proof of Theorem 1. In section 3, we will present the proof of Theorem 1 and using that we will find an upper bound in terms of t for the number of edges of an r-graph to be able to covered by t perfect matchings. In Section 4, we will present a new conjecture that generalizes the idea presented by Patel in [8], that might be useful to approach Generalized Berge Conjecture.

2 The perfect matching polytope

Let $G = (V, E)$ be a graph which may contain multiple edges. For any set $C \subseteq E(G)$, if $G - C$ has more components than G, then C is said to be an *edge-cut* in G. A k-edge-cut is an edge-cut consists of k edges. Recall that $\partial(X)$ is defined as the set of edges with precisely one end in $X \subseteq V(G)$. An edge-cut C is *odd* if there exists $X \subseteq V(G)$ of odd cardinality such that $C = \partial(X)$. Notice that if G is an r-graph and $X \subseteq V(G)$ is an odd cardinality set, then r and $|\partial(X)|$ have the same parity.

Let w be a vector in $\mathbb{R}^{E(G)}$. The entry of w corresponding to an edge e is denoted by $w(e)$, and for $A \subseteq E(G)$, we define $w(A) = \sum_{e \in A} w(e)$. The vector w is a fractional perfect matching of G if it satisfies the following properties:

i. $0 \leq w(e) \leq 1$ for each $e \in E(G)$,

ii. $w(\partial(v)) = 1$ for each vertex $v \in V(G)$,

iii. $w(\partial(X)) \geq 1$ for each $X \subseteq V(G)$ of odd cardinality.

Note that $w = (\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r})$ is a fractional perfect matching for any r-graph G.

Let $P(G)$ denote the set of all fractional perfect matchings of G. Clearly, if M is a perfect matching, then the characteristic vector χ^M of M is contained in $P(G)$. Also, if $w_1, \ldots, w_n \in$

$P(G)$, then any convex combination, $\sum_i \lambda_i w_i$ with $0 \le \lambda_i \le 1$ such that $\sum_i \lambda_i = 1$, of them belongs to $P(G)$. So $P(G)$ contains the convex hull of all vectors χ^M such that M is a perfect matching of G. The Perfect Matching Polytope Theorem of Edmonds [1] asserts that the converse inclusion also holds:

Theorem 2 (Perfect Matching Polytope Theorem)**.** *For any graph G, the set $P(G)$ is precisely the convex hull of the characteristic vectors of perfect matchings of G.*

The following lemma deducted from Edmonds' Perfect Matching Polytope Theorem, which introduced by Kaiser, Král, and Norine in [4], plays a crucial role in our result.

Lemma 1. *[4] If w is a fractional perfect matching in a graph G and $c \in \mathbb{R}^E$, then G has a perfect matching M such that $c \cdot \chi^M \ge c \cdot w$, where $\cdot$ denotes the dot product . Moreover M contains exactly one edge of each odd cut C with $w(C) = 1$.*

3 Proof of Theorem 1

Let $\mathcal{M}^t = \{M_1, \ldots, M_t\}$ be a set of $t \ge 0$ perfect matchings in an r-graph G. Recall that for any positive integers $r \ge 3$ and $t \ge 0$, we define

$$m_t^r = \inf_G \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}$$

where the infimum is taken over all r-graphs. It is clear that $m_0^r = 0$ and $m_1^r = \frac{1}{r}$. For any fixed integer r we define $a_0^r = 0$ and for $t \ge 1$,

$$a_t^r = a_{t-1} + (1 - a_{t-1}) \frac{2 + (t-1)\left(r^2 - r - 4\right)}{2r + (t-1)\left(r^3 - r^2 - 6r + 4\right)}.$$

We will show that $m_t^r \ge a_t^r$ for each index t and fixed $r \ge 3$. Before presenting the proof let us give some definitions. Let G be an r-graph and $\mathcal{M}^t = \{M_1, \ldots, M_t\}$ be a set of t perfect matchings in G for $t \ge 0$. For each subset $A \subseteq E(G)$, we define

$$\Phi(A, \mathcal{M}^t) = \sum_{i=1}^{t} |A \cap M_i|.$$

We further define the function $w_t^r : E(G) \to \mathbb{R}$ for any fixed integers r and t as;

$$w_t^r(e) = \frac{2 + t\left(r^2 - r - 4\right) - 2\left(r-2\right)\Phi(e, \mathcal{M}^t)}{2r + t\left(r^3 - r^2 - 6r + 4\right)}.$$

We would like to remark that the recursive formula a_t^r and the function $w_t^r(e)$ defined above are exactly the generalizations of the ones given in [7] for cubic graphs.

Observe that when $t = 0$, we have $\mathcal{M}^0 = \emptyset$ and $\Phi(\{e\}, \mathcal{M}^0) = 0$ for each $e \in E(G)$. Hence $w_0^r(e) = \frac{1}{r}$ which is a fractional perfect matching. We further note that Kaiser, Král, and Norine [4], used w_1^3 and w_2^3 in their proof for $m_2^3 = \frac{3}{5}$ and $\frac{27}{35} \le m_3^3 \le \frac{4}{5}$.

A natural extension of the function w_t^r for a set $A \subseteq E(G)$ is defined as;

$$w_t^r(A) = \sum_{e \in A} w_t^r(e) = \frac{|A| \cdot \left[2 + t\left(r^2 - r - 4\right)\right] - 2\,(r-2)\,\Phi(A, \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)}.$$

Instead of proving Theorem 1 directly, we prove the following technical theorem which is slightly stronger.

Theorem 3. *For any r-graph G with $r \geq 3$ and any integer $t \geq 0$, there exists a set of t perfect matchings $\mathcal{M}^t = \{M_1, M_2, \ldots, M_t\}$ such that*

(i) the function $w_t^r : E(G) \to \mathbb{R}$ defined as $w_t^r(e) = \frac{2+t\left(r^2-r-4\right)-2(r-2)\Phi(e,\mathcal{M}^t)}{2r+t(r^3-r^2-6r+4)}$ is a fractional perfect matching, and

(ii)

$$\frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|} \geq a_t^r,$$

which consequently yields $m_t^r \geq a_t^r$.

Proof: Let G be an r-graph. We prove *(i)* and *(ii)* simultaneously by induction on t.

For $t = 0$, let $\mathcal{M}^0 = \emptyset$. Then by the definition of $w_t^r(e)$, we have $w_0^r(e) = \frac{2}{2r} = \frac{1}{r}$ for any $e \in E(G)$ and as observed earlier w_0^r is a fractional perfect matching. Therefore (i) follows. Since by definition $a_0^r = 0$, *(ii)* holds trivially. Now suppose that $t \geq 1$ and let $\mathcal{M}^{t-1} = \{M_1, \ldots, M_{t-1}\}$ be a set of $t-1$ perfect matchings in G such that w_{t-1}^r is a fractional perfect matching. Let $c = 1 - \chi^{\cup_{i=1}^{t-1} M_i}$. By Lemma 1 there exists a perfect matching, M_t, in G such that $c \cdot \chi^{M_t} \geq c \cdot w_{t-1}^r$ and M_t contains exactly one edge of each odd cut $\partial(X)$ with $|X|$ odd and $w_{t-1}^r(C) = 1$.

In order to prove that $w_t^r(e)$ is a fractional perfect matching, we need to verify the three conditions in the definition of fractional perfect matching.

(a) for each edge $e \in E(G)$, it is clear that $\Phi(e, \mathcal{M}^t) \geq 0$ and therefore

$$w_t^r(e) = \frac{2 + t\left(r^2 - r - 4\right) - 2\,(r-2)\,\Phi(e, \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \leq \frac{2 + t\left(r^2 - r - 4\right)}{2r + t\,(r^3 - r^2 - 6r + 4)}.$$

It is easy to verify that $r^3 - r^2 - 6r + 4 \geq r^2 - r - 4 > 0$ for all $r \geq 3$, so we have $w_t^r(e) \leq 1$. Moreover, since $\Phi(e, \mathcal{M}^t) = \sum_{i=1}^{t} |\{e\} \cap M_i| \leq t$, we have

$$w_t^r(e) \geq \frac{2 + t\left(r^2 - r - 4\right) - 2\,(r-2)\,t}{2r + t\,(r^3 - r^2 - 6r + 4)} = \frac{2 + t(r^2 - 3r)}{2r + t\,(r^3 - r^2 - 6r + 4)}.$$

Note that $r^2 - 3r$ and $r^3 - r^2 - 6r + 4$ are positive for all $r \geq 3$. Hence $w_t^r(e) \geq 0$. Therefore $0 \leq w_t^r(e) \leq 1$.

(b) Let $v \in V(G)$ be a vertex. It is clear that $|\partial(v) \cap M| = 1$ for any perfect matching M. Therefore $\Phi(\partial(v), \mathcal{M}^t) = \sum_{i=1}^{t} |\partial(v) \cap M_i| = t$. This together with $|\partial(v)| = r$ gives us

$$\begin{aligned} w_t^r(\partial(v)) &= \frac{2r + tr\left(r^2 - r - 4\right) - 2\,(r-2)\,\Phi(\partial(v), \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2r + t\left(r^3 - r^2 - 4r\right) - 2\,(r-2)\,t}{2r + t\,(r^3 - r^2 - 6r + 4)} = 1 \end{aligned}$$

as required.

(c) Let X be an odd subset of $V(G)$ with $|\partial(X)| = k$. Since G is an r-graph, we have $k \geq r$ and note that k and r have the same parity. By induction hypothesis we have,

$$w_{t-1}^r(\partial(X)) = \frac{2k + (t-1)k\left(r^2 - r - 4\right) - 2\,(r-2)\,\Phi(\partial(X), \mathcal{M}^{t-1})}{2r + (t-1)\,(r^3 - r^2 - 6r + 4)} \geq 1. \qquad (1)$$

We will show that $w_t^r(\partial(X)) \geq 1$ by considering three cases.

Case 1: $k = r$

From inequality (1) we get $\Phi(\partial(X), \mathcal{M}^{t-1}) \leq t-1$. On the other hand, each r-cut intersects each of the $t-1$ perfect matchings $M_1, M_2, \ldots M_{t-1}$ at least once, which yields $\Phi(\partial(X), \mathcal{M}^{t-1}) \geq t-1$. Hence $\Phi(\partial(X), \mathcal{M}^{t-1}) = t-1$, and so

$$w_{t-1}^r(\partial(X)) = \frac{2r + (t-1)\left(r^3 - r^2 - 4r\right) - 2\,(r-2)\,(t-1)}{2r + (t-1)\,(r^3 - r^2 - 6r + 4)} = 1.$$

Then by the choice of M_t, we conclude that $|\partial(X) \cap M_t| = 1$ from Lemma 1. Therefore

$$\begin{aligned} w_t^r(\partial(v)) &= \frac{2r + tr\left(r^2 - r - 4\right) - 2\,(r-2)\,\Phi(\partial(X), \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2r + t\left(r^3 - r^2 - 4r\right) - 2\,(r-2)\left[\Phi(\partial(X), \mathcal{M}^{t-1}) + |\partial(X) \cap M_t|\right]}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2r + t\left(r^3 - r^2 - 4r\right) - 2\,(r-2)\,t}{2r + t\,(r^3 - r^2 - 6r + 4)} = 1. \end{aligned}$$

Case 2: $k > r$, $w_{t-1}^r(\partial(X)) = 1$

First, note that since $w_{t-1}^r(\partial(X)) = 1$, by Lemma 1 we have $|\partial(X) \cap M_t| = 1$.

We will show that

$$\begin{aligned} w_t^r(\partial(X)) &= \frac{2k + tk\left(r^2 - r - 4\right) - 2\,(r-2)\,\Phi(\partial(X), \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2k + tk\left(r^2 - r - 4\right) - 2(r-2)\left[\Phi(\partial(X), \mathcal{M}^{t-1}) + |\partial(X) \cap M_t|\right]}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2k + tk\left(r^2 - r - 4\right) - 2(r-2)\left[\Phi(\partial(X), \mathcal{M}^{t-1}) + 1\right]}{2r + t\,(r^3 - r^2 - 6r + 4)} \geq 1, \end{aligned}$$

Since $w_{t-1}^r(\partial(X)) = \frac{2k+(t-1)k\left(r^2-r-4\right)-2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1})}{2r+(t-1)(r^3-r^2-6r+4)} = 1$, we have

$$2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1}) = 2k - 2r + (t-1)k\left(r^2-r-4\right) - (t-1)\left(r^3-r^2-6r+4\right).$$

Substituting and simplifying gives,

$$w_t^r(\partial(X)) \quad = \quad \frac{2r + t\left(r^3-r^2-6r+4\right) + A}{2r + t\left(r^3-r^2-6r+4\right)},$$

where $A = k\left(r^2-r-4\right) - \left(r^3-r^2-6r+4\right) - 2\left(r-2\right)$. In order to see that $w_t^r(\partial(X)) \geq 1$, it is enough to show that $A \geq 0$. This holds because

$$A = k\left(r^2-r-4\right) - (r^3-r^2-6r+4) - 2(r-2) = (k-r)(r^2-r-4),$$

and $r^2-r-4 > 0$ for all $r \geq 3$. This completes the proof of the case 2.

Case 3: $k > r$, $w_{t-1}^r(\partial(X)) > 1$

Since $w_{t-1}^r(\partial(X)) = \frac{2k+(t-1)k\left(r^2-r-4\right)-2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1})}{2r+(t-1)(r^3-r^2-6r+4)} > 1$, we have

$$\begin{aligned} 2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1}) &< 2k - 2r + (t-1)\left[k(r^2-r-4) - (r^3-r^2-6r+4)\right] \\ &= 2(k-r) + (t-1)\left[(k-r)(r^2-r-4) + 2(r-2)\right]. \end{aligned} \tag{2}$$

Notice that because $k-r$ is even, both sides of inequality (2) is even. Hence we have

$$2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1}) \leq 2(k-r) + (t-1)\left[(k-r)(r^2-r-4) + 2(r-2)\right] - 2. \tag{3}$$

Now we will show that

$$w_t^r(\partial(X)) = \frac{2k + tk\left(r^2-r-4\right) - 2(r-2)\left[\Phi(\partial(X),\mathcal{M}^{t-1}) + |\partial(X) \cap M_t|\right]}{2r + t\left(r^3-r^2-6r+4\right)} \geq 1.$$

Applying $|\partial(X) \cap M_t| \leq k$ and by inequality (3), we obtain

$$w_t^r(\partial(X)) \geq \frac{2r + t\left(r^3-r^2-6r+4\right) + B}{2r + t\left(r^3-r^2-6r+4\right)},$$

where $B = k\left(r^2-r-4\right) - (r^3-r^2-6r+4) + 2 - 2(r-2)k = (k-r)(r^3-3r) + 2(r-1)$. Since $r \geq 3$ and $k > r$, we have $B > 0$. Hence $w_t^r(\partial(X) \geq 1$, and we are done with the last case. Therefore the function $w_t^r(e)$ is a fractional perfect matching for any integer $t \geq 0$ and fixed $r \geq 3$.

We now complete the prove of *(ii)*. The assertion is clearly true for $t = 1$ as $a_1^r = \frac{1}{r}$. So we may assume $t \geq 2$. By induction hypothesis, we have

$$\frac{\left|\bigcup_{i=1}^{t-1} M_i\right|}{E(G)} \geq a_{t-1}^r.$$

Recall $c = 1 - \chi^{\bigcup_{i=1}^{t-1} M_i}$. By the choice of M_t, we have

$$c \cdot \chi^{M_t} \geq c \cdot w_{t-1}^r. \tag{4}$$

Here the left hand side of (4) is $c \cdot \chi^{M_t} = |M_t \setminus \bigcup_{i=1}^{t-1} M_i|$. Since for each edge $e \notin \bigcup_{i=1}^{t-1} M_i$, we have $w_{t-1}^r(e) = \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)(r^3-r^2-6r+4)}$. So the right hand side of (4) is the number of edges not covered by $\mathcal{M}^{t-1}$ multiplied by $\frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)(r^3-r^2-6r+4)}$ which gives

$$|M_t \setminus \bigcup_{i=1}^{t-1} M_i| \geq \left(|E(G)| - |\bigcup_{i=1}^{t-1} M_i|\right) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)}.$$

Hence

$$\begin{aligned} |\bigcup_{i=1}^{t} M_i| &= |M_t \setminus \bigcup_{i=1}^{t-1} M_i| + |\bigcup_{i=1}^{t-1} M_i| \\ &\geq \left(|E(G)| - |\bigcup_{i=1}^{t-1} M_i|\right) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)} + |\bigcup_{i=1}^{t-1} M_i|. \end{aligned}$$

Dividing by $|E(G)|$ gives

$$\frac{|\bigcup_{i=1}^{t} M_i|}{|E(G)|} \geq \left(1 - \frac{|\bigcup_{i=1}^{t-1} M_i|}{|E(G)|}\right) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)} + \frac{|\bigcup_{i=1}^{t-1} M_i|}{|E(G)|}.$$

With the assumption that $\frac{|\bigcup_{i=1}^{t-1} M_i|}{|E(G)|} \geq a_{t-1}$, we conclude that

$$\frac{|\bigcup_{i=1}^{t} M_i|}{|E(G)|} \geq a_{t-1}^r + (1 - a_{t-1}^r) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)} = a_t^r.$$

Since by definition, $m_t^r = \inf_G \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}$, we have

$$m_t^r \geq a_t^r,$$

for any integer $t \geq 0$ and fixed $r \geq 3$. $\square$

3.1 Covering an r- graph with t Perfect Matchings

It is still unknown whether $m_t^r = 1$ for any $r \geq 3$ and $t \geq 2r-1$. The best known result for $r = 3$ is given by Mazzuoccolo which states that: if a cubic bridgeless graph G has fewer than $\lfloor \frac{2^t}{\sqrt{t}} \rfloor$ edges, then there is a covering of G by t perfect matchings. Now we will generalize his result and provide an upper bound for the size of an r-graph G so that G can be covered by t perfect matchings by using Theorem 3.

Theorem 4. *Let G is an r-graph and t be a positive integer. If $|E(G)| < \frac{1}{\sqrt{t}} \left(\frac{r^3-r^2-6r+4}{r^3-2r^2-5r+8} \right)^t$, then G can be covered by t perfect matchings.*

Proof. As mentioned earlier, the special case $r = 3$ in Theorem 4 was proved in [7]. Therefore in the proof it is enough for us to consider the case $r \geq 4$ and $t \geq 2$. Fix $r \geq 4$. Note that if $|E(G)| \cdot m_t^r > |E(G)| - 1$, then there exists a covering of G by t perfect matchings. In other words, if $|E(G)| < \frac{1}{1-m_t^r}$ then we have a covering of $E(G)$ by t perfect matchings. By theorem 3, we know that $m_t^r \geq a_t^r$, that is $\frac{1}{1-m_t^r} \geq \frac{1}{1-a_t^r}$. So it is enough to show that $\frac{1}{\sqrt{t}} \left(\frac{r^3-r^2-6r+4}{r^3-2r^2-5r+8} \right)^t \leq \frac{1}{1-a_t^r}$, or equivalently $a_t^r \geq 1 - \sqrt{t} \left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \right)^t$ for each $t \geq 2$. We prove by induction on t. For the base case, when $t = 2$,

$$\begin{aligned} a_2^r &= \frac{r+1}{r^2+r-2} + \frac{1}{r}\left(1 - \frac{r+1}{r^2+r-2}\right) \\ &= \frac{2r+3}{r(r+2)}. \end{aligned}$$

Now we want to show that the following inequality holds for all $r \geq 4$:

$$\frac{2r+3}{r(r+2)} \geq 1 - \sqrt{2} \left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \right)^2. \tag{5}$$

First note that $a_2^r \geq 0$, for all $r \geq 4$ and one can easily check that inequality 5 holds for $4 \leq r \leq 6$. Let $f(r) := 1 - \sqrt{2} \left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \right)^2$. Then

$$f'(r) = -\frac{2^{\frac{3}{2}} \left(r^3 - 2r^2 - 5r + 8\right) \left(r^4 - 2r^3 - 5r^2 + 28\right)}{\left(r^3 - r^2 - 6r + 4\right)^3}.$$

It is easy to see that $f'(r) \leq 0$ for all $r \geq 6$. Therefore $f(r)$ is decreasing and $f(7) \leq 0$. Hence we conclude inequality (5) holds for all $r \geq 4$ and so the result follows for $t = 2$.

Now suppose $t \geq 3$ and $a_t^r \geq 1 - \sqrt{t} \left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \right)^t$ for each $r \geq 4$. We will show that

$$a_{t+1}^r \geq 1 - \sqrt{t+1}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^{t+1}.$$

For the left hand side, we have

$$a_{t+1}^r = \frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)} + a_t^r \cdot \left(1 - \frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)}\right).$$

Applying the induction hypothesis gives,

$$\begin{aligned} a_{t+1}^r &\geq \frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)} + \\ &\quad \left(1-\sqrt{t}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^t\right)\cdot\left(1-\frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)}\right) \\ &= 1-\sqrt{t}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^t \cdot \frac{2r-2+t\left(r^3-2r^2-5r+8\right)}{2r+t\left(r^3-r^2-6r+4\right)} = D. \end{aligned}$$

Now we are done if we can show that

$$D \geq 1-\sqrt{t+1}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^{t+1},$$

or simply

$$\frac{2r-2+t\left(r^3-2r^2-5r+8\right)}{2r+t\left(r^3-r^2-6r+4\right)} \leq \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\sqrt{1+\frac{1}{t}}. \tag{6}$$

For the left hand side of (6) we have

$$\begin{aligned} \frac{2r-2+t\left(r^3-2r^2-5r+8\right)}{2r+t\left(r^3-r^2-6r+4\right)} &\leq \frac{2r+t\left(r^3-2r^2-5r+8\right)}{t\left(r^3-r^2-6r+4\right)} \\ &= \frac{2r}{t\left(r^3-r^2-6r+4\right)} + \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \end{aligned}$$

For the right hand side of (6), we have

$$\sqrt{1+\frac{1}{t}} = 1+\frac{1}{2t}-\frac{1}{8t^2}+\frac{1}{16t^3}-\frac{5}{128t^4}+\dots$$

from the binomial expansion, which leads to

$$\sqrt{1+\frac{1}{t}} \geq 1+\frac{1}{2t}-\frac{1}{8t^2}.$$

Hence the right hand side of inequality (6) has the following lower bound:

$$\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\sqrt{1+\frac{1}{t}} \geq \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\left(1+\frac{1}{2t}-\frac{1}{8t^2}\right),$$

So for the inequality (6), it is enough to show that

$$\frac{2r}{t\left(r^3-r^2-6r+4\right)}+\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \leq \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\left(1+\frac{1}{2t}-\frac{1}{8t^2}\right),$$

which can be simplified to

$$\frac{16t}{4t-1} \leq r^2-2r-5+\frac{8}{r}.$$

One can easily check that the last inequality holds for any $r \geq 4$ and $t \geq 2$. Therefore we proved $a^r_t \geq 1-\sqrt{t}\left(\frac{r^2-3r+1}{r^2-2r-1}\right)^t$ for each $t \geq 2$ and we are done. □

Theorem 4 gives an upper bound, in terms of t, for the number of edges of an r-graph G so that G can be covered by t perfect matchings. Here we want to note a trivial upper bound for t such that any r-graph G has a covering by t perfect matching.

Seymour, in [10], generalized the well-known Petersen Theorem, which states that every cubic bridgeless graph (3-graph) has a perfect matching, to r-graphs. Therefore we know that every r-graph G has at least one perfect matching, and according to the Tutte's Theorem, every edge of an r-graph G is contained in at least one perfect matching. Since $|E(G)| = \frac{nr}{2}$, trivially there is a family of $\frac{nr}{2}$ perfect matchings of G (not necessarily distinct) that covers $E(G)$. That is $m^r_{\frac{nr}{2}}(G) = 1$, for any $r \geq 3$.

Now we turn our attention to r-graphs where $r \geq \frac{|V(G)|}{2}$. We know now by Theorem 4 if

$$\frac{nr}{2} < \frac{1}{\sqrt{t}}\left(\frac{r^3-r^2-6r+4}{r^3-2r^2-5r+8}\right)^t, \tag{7}$$

then G can be covered by t perfect matchings.

Taking the log of both sides in (7) gives

$$\log(n)+\log(r)-\log(2) < t\log\left(1+\frac{r^2-r-4}{r^3-2r^2-5r+8}\right)-\frac{1}{2}\log(t),$$

which yield

$$t > \frac{\log(n) + \log(r) - \log(2) + \frac{1}{2}\log(t)}{\log\left(1 + \frac{r^2 - r - 4}{r^3 - 2r^2 - 5r + 8}\right)}. \tag{8}$$

Here we note that $f(r) := \log\left(1 + \frac{r^2-r-4}{r^3-2r^2-5r+8}\right) - 1/r > 0$. Indeed

$$f'(r) = -\frac{r^5 + 4r^4 - 29r^3 + 14r^2 + 68r - 32}{r^2 \cdot (r^3 - 2r^2 - 5r + 8)\,(r^3 - r^2 - 6r + 4)} < 0,$$

for all $r \geq 3$. Moreover $f(3) = 0.3598 > 0$, and $\lim_{r\to\infty} f(r) = 0$.

Therefore if

$$t > \frac{\log n + \log r - \log 2 + \frac{1}{2}t}{\frac{1}{r}},$$

then G can be covered by t perfect matchings. Since $n \leq 2r$ by assumption and $t \leq \frac{nr}{2}$, taking

$$t = \lceil 3r \log r \rceil$$

gives $m_t^r(G) = 1$.

Corollary 3.1. *Let G be an r-graph of order $n \leq 2r$. Then $m_t^r(G) = 1$ when $t = \lceil 3r \log r \rceil$.*

4 A New Conjecture

In this section we will give a natural generalization of Conjecture 1.3 for any $r \geq 3$, and using that we generalize the results given in [8] by Patel. We further present a new conjecture that may help in the proof of Generalized Fulkerson Conjecture.

First we define

$$\tau_t^r = \frac{t(4r - t - 1)}{2r(2r - 1)},$$

for any $r \geq 3$ and $t \geq 0$. Note that $\tau_t^3 = m_t^r(P)$, where P is the Petersen Graph.

Remark 2. *Let G be an r-graph having $2r$ perfect matchings, $M_1, \ldots, M_{2r}$, such that $|M_i \cap M_j| = 1$ for each $i \neq j$, and for each $e \in E(G)$ there is a unique pair of perfect matchings M_i and M_j so that $e \in M_i \cap M_j$. Note that for any r-graph satisfying above property, we have $m_t^r(G) = \tau_t^r$ for any fixed $1 \leq t \leq 2r - 1$.*

Conjecture 4.1. *$m_t^r \geq \tau_t^r$ for $1 \leq t \leq 2r - 1$. Specifically, $m_{2r-1}^r = 1$.*

The property explained in Remark 2 clearly holds for the 3-graphs; the Petersen graph. However there is no r-graph known satisfying that property for $r > 3$ as far as we know to this date. If one can find such an r-graph G among all r-graphs, then $\tau_t^r = m_t^r(G) \geq m_t^r$. Hence Conjecture 4.1 implies $m_t^r = \tau_t^r$ and the following theorem still holds. Now we show that Conjecture 1.5 implies Conjecture 4.1.

Theorem 5. *Generalized Fulkerson Conjecture (GFC) implies Conjecture 4.1.*

Proof. It suffices to show that for any r-graph G and each $1 \leq t \leq 2r-1$, $m_t(G) \geq \tau_t^r$. Fix $1 \leq t \leq 2r-1$. Given GFC holds for G, we can find a set of $2r$ perfect matchings, $\mathcal{M} = \{M_1, \ldots, M_{2r}\}$, such that each edge of G is contained in exactly two elements in $\mathcal{M}$.

Let S_t be a set of t elements chosen uniformly and randomly from $[2r]$. Fix $e \in E(G)$. Since GFC holds for G, there exists two perfect matchings, say M_a and M_b, in $\mathcal{M}$ that contains e. Then

$$\begin{aligned}\mathbb{P}(e \in \bigcup_{i \in S_t} M_i) &= \mathbb{P}(a \in S_t \text{ or } b \in S_t) \\ &= 1 - \mathbb{P}(a \notin S_t \text{ and } b \notin S_t) \\ &= 1 - \frac{\binom{2r-2}{t}}{\binom{2r}{t}} \\ &= \tau_t^r.\end{aligned}$$

Further we have the expectation,

$$\mathbb{E}(|\bigcup_{i \in S_t} M_i|) = \sum_{e \in E(G)} \mathbb{P}(e \in \bigcup_{i \in S_t} M_i) = |E(G)| \cdot \tau_t^r.$$

Therefore, there exists some t-element subset of $[2r]$, say S_t^*, satisfying

$$|\bigcup_{i \in S_t^*} M_i| \geq |E(G)| \cdot \tau_t^r.$$

Hence $m_t(G) \geq \tau_t^r$.

□

Now we will give a conjecture that is stronger than Conjecture 4.1.

Conjecture 4.2. *Let G be an r-graph. For each $t \in \{1, \ldots 2r-1\}$, G has t perfect matchings, $M_1, \ldots, M_t$, satisfying:*

1. *no edge of G is contained in more than two of the M_i's,*
2. *$|\bigcup_{i=1}^t M_i| \geq \tau_t^r \cdot |E(G)|$, and*
3. *for every odd cut C of G, if $|C| = k$ then $\sum_{i=1}^t |M_i \cap C| \leq 2(k-r)+t$.*

We will show later that GFC implies Conjecture 4.2, but let us first present the reason why Conjecture 4.2 could be useful for proving Conjecture 4.1.

Theorem 6. *If Conjecture 4.2 holds for a given $t \in \{2, \ldots 2r-2\}$, then Conjecture 4.1 holds for $t+1$. If Conjecture 4.2 holds for $t = 2r-1$, then GFC holds.*

Proof. Let G be an r-graph. Suppose G has t perfect matchings, $M_1, \ldots, M_t$ satisfying Conjecture 4.2 for $t \in \{2 \ldots, 2r-2\}$. Then set

$$w_t(e) = \begin{cases} 0 & \textit{if} \quad e \text{ is in exactly two of } M_1, \ldots M_t; \\ \frac{1}{2r-t} & \textit{if} \quad e \text{ is in exactly one of } M_1, \ldots M_t; \\ \frac{2}{2r-t} & \textit{if} \quad e \text{ is not in any of } M_1, \ldots M_t. \end{cases}$$

Now we will check $w_t(e)$ is a fractional perfect matching for any $t \in \{2, \ldots 2r-2\}$ by checking the three condition given in the definition of fractional perfect matching.

i. Since $2 \leq t \leq 2r-2$, clearly $0 \leq w_t(e) \leq 1$.

ii. For any $v \in V(G)$, let a_0, a_1 and a_2 denote the number of edges of $\partial(v)$ that are covered by no, exactly one and exactly 2 perfect matchings respectively. Note that

$$a_0 + a_1 + a_2 = r \tag{9}$$

Also since $|M_i \cap \partial(v)| = 1$ for all $1 \leq i \leq t$, we have

$$a_1 + 2a_2 = t \tag{10}$$

Taking $\frac{2}{2r-t}$ times (relation (9)) $-$ $\frac{1}{2r-t}$ times (relation (10)) gives

$$w_t(\partial(v)) = \frac{2}{2r-t}a_0 + \frac{1}{2r-t}a_1 = 1.$$

So this condition is satisfied.

iii. Let $X \subseteq V(G)$ be an odd cardinality set with $|\partial(X)| = k$. Since G is an r-graph, it follows that $k \geq r$. Let b_0, b_1 and b_2 denote the number of edges of $\partial(X)$ that are covered by no, exactly one and exactly 2 perfect matchings respectively. Note that

$$b_0 + b_1 + b_2 = k, \tag{11}$$

and by Conjecture 4.2 we have

$$b_1 + 2b_2 \leq 2(k-r) + t. \tag{12}$$

Taking $\frac{2}{2r-t}$ times (relation (11)) $-$ $\frac{1}{2r-t}$ times (relation (12)) gives

$$w_t(\partial(X)) \geq \frac{2}{2r-t}b_0 + \frac{1}{2r-t}b_1 \geq 1.$$

as we wanted and third condition is also satisfied. Hence $w_t(e)$ is a fractional perfect matching. By Lemma 1, there exists a perfect matching, say M_{t+1}, such that c $\cdot\chi^{M_{t+1}} \geq c \cdot w_t(e)$. Setting $c = \chi^{(\bigcup_{i=1}^{t} M_i)^c}$ yields

$$|M_{t+1} \setminus \bigcup_{i=1}^{t} M_i)| = \chi^{(\bigcup_{i=1}^{t} M_i)^c} \cdot \chi^{M_{t+1}} \geq \chi^{(\bigcup_{i=1}^{t} M_i)^c} \cdot w_(e) = \frac{2}{2r-t} \cdot |E(G) \setminus \bigcup_{i=1}^{t} M_i|.$$

Therefore

$$\begin{aligned}
|\bigcup_{i=1}^{t+1} M_i| &= |\bigcup_{i=1}^{t} M_i| + |M_{t+1} \setminus \bigcup_{i=1}^{t} M_i| \\
&\geq |\bigcup_{i=1}^{t} M_i| + \frac{2}{2r-t} \cdot |E(G) \setminus \bigcup_{i=1}^{t} M_i| \\
&= \frac{2r-t-2}{2r-t} \cdot |\bigcup_{i=1}^{t} M_i| + \frac{2}{2r-t} \cdot |E(G)| \\
&\geq \frac{2r-t-2}{2r-t} \cdot m_t^r |E(G)| + \frac{2}{2r-t} \cdot |E(G)| \\
&= (\frac{2r-t-2}{2r-t} \cdot \frac{t(4r-t-1)}{2r(2r-1)} + \frac{2}{2r-t}) \cdot |E(G)| \\
&= \tau_{t+1}^r \cdot |E(G)|
\end{aligned}$$

Thus G satisfies Conjecture 4.1 for $2 \leq t \leq 2r-1$. Note that when $t = 2r-1$, then GFC holds.

□

Theorem 7. *The GFC implies Conjecture 4.2.*

Proof. Let G be an r-graph satisfying GFC, that is G has $2r$ perfect matchings, $M_1, \dots, M_{2r}$ with each edge of G are in exactly two of them. Clearly, for each $t \in \{2, \dots, 2r-1\}$, any t-subset of $\{M_1, \dots, M_{2r}\}$ satisfy the first condition of the Conjecture 4.2.

By the Theorem 5, we know that GFC implies $m_t^r \geq \tau_t^r$. Since $m_t^r = \inf_G \max_{M_1,\dots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}$ where the infimum taken over all r-graphs, we have

$$\tau_t^r \cdot |E(G)| \leq m_t^r \cdot |E(G)| \leq |\bigcup_{i=1}^{t} M_i|.$$

So the second condition of Conjecture 4.2 is also satisfied.

For the third condition, first note that for any perfect matching M and any odd cut C then $|C \cap M| \geq 1$. Let $|C| = k$. Since $\sum_{i=1}^{2r} |C \cap M_i| = 2k$, then for $S \subseteq [2r]$ with $|S| = t$, we have

$$\begin{aligned}
\sum_{i \in S} |M_i \cap C| &= 2k - \sum_{i \notin S} |M_i \cap C| \\
&\leq 2k - |[2r] \setminus S| \\
&= 2k - (2r-t) \\
&= 2(k-r) + t
\end{aligned}$$

as required. ☐

References

[1] J. Edmonds. Maximum Matching and a Polyhedron with 0, 1-vertices. *Journal of Research of the National Bureau of Standards. Section B. Mathematics and Mathematical Physics*, 69:125–130, 1965.

[2] D. R. Fulkerson. Blocking and Anti-blocking Pairs of Polyhedra. *Mathematical programming*, 1(1):168–194, 1971-12.

[3] L. Jin and E. Steffen. Unions of 1-factors in r-Graphs and Overfull Graphs. *Journal of Combinatorics*, 11(3):457–473, 2020.

[4] T. Kaiser, D. Král, and S. Norine. *Unions of Perfect Matchings in Cubic Graphs*. Springer, Berlin ; New York, 2006.

[5] G. Mazzuoccolo. The Equivalence of Two Conjectures of Berge and Fulkerson. *Journal of Graph Theory.*, 68(2):125–128, 2011-10.

[6] G. Mazzuoccolo. An Upper Bound for the Excessive Index of an r-Graph. *Journal of Graph Theory.*, 73(4):377–385, 2013-08.

[7] G. Mazzuoccolo. Covering a Cubic Graph with Perfect Matchings. *Discrete Mathematics*, 313(20):2292–2296, 2013-10.

[8] V. Patel. Unions of Perfect Matchings in Cubic Graphs and Implications of the Berge-Fulkerson Conjecture. *DAM Research Report, LSE-CDAM-2006-06*, 01 2006.

[9] J. Petersen. Die Theorie der Regulären Graphs. *Acta Mathematica*, 15:193 – 220, 1900.

[10] P. D. Seymour. On Multi-Colourings of Cubic Graphs, and Conjectures of Fulkerson and Tutte. *Proceedings of the London Mathematical Society*, s3-38(3):423–460, 05 1979.

[11] W. T. Tutte. The Factorization of Linear Graphs. *Journal of the London Mathematical Society*, s1-22(2):107–111, 1947.